AMBIGUOUS CLASS NUMBER FORMULAS

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Abstract. An elementary proof of Chevalley’s ambiguous class number formula is presented.

1. Introduction

In Gras’ book [2, p. 178, p. 180] one finds Chevalley’s ambiguous class formulas. In Lemmermeyer [3] one finds a modern and elementary proof. This Note gives a different elementary proof of this result, which uses basic results proved in Lang’s book [1].

Let $K/k$ be a cyclic extension of number fields with Galois group $G = \text{Gal}(K/k) = \langle \sigma \rangle$, where $\sigma$ is a generator of $G$. Denote by $\mathfrak{D}$ and $\mathfrak{D}_{\infty}$ the ring of integers of $k$ and $K$, respectively. Let $\infty$ and $\infty_r$ (resp. $\tilde{\infty}$ and $\tilde{\infty}_r$) denote the set of infinite and real places of $k$ (resp. of $K$), respectively, and $A_k$ (resp. $A_K$) the adele ring of $k$ (resp. $K$). We shall identify a real cycle $c$ with its support, which is a subset of real places. Let $r_k : \tilde{\infty} \to \infty$ denote the restriction to $k$.

Let $\tilde{c}$ be a real cycle on $K$ which is stable under the $G$-action. Denote by

$$\text{Cl}(K, \tilde{c}) := \frac{A_K^\times}{K^\times \mathfrak{D}^\times K_\infty^\times(\tilde{c})^\times},$$

the narrow ideal class group of $K$ with respect to $\tilde{c}$, where $\mathfrak{D}$ is the profinite completion of $\mathfrak{D}$, and $K_\infty^\times(\tilde{c})^\times = \{ a = (a_w) \in K_\infty^\times \mid a_w > 0 \ \forall \ w \in \tilde{c} \}$. Similarly one defines $\text{Cl}(k, c)$ for any real cycle $c$ on $k$. The group $G$ acts on the finite abelian group $\text{Cl}(K, \tilde{c})$. Its $G$-invariant subgroup $\text{Cl}(K, \tilde{c})^G$ is called the ambiguous ideal class group (with respect to $\tilde{c}$).

Let $\epsilon$ be the real cycle on $k$ such that $\infty_r - \epsilon = r_k(\tilde{\infty}_r - \tilde{\epsilon})$ and $\epsilon_0 := r_k(\tilde{\epsilon})$. One has $\epsilon = \epsilon_0 \infty^c_\epsilon$, where $\infty^c_\epsilon$ is the set of real places of $k$ which does not split completely in $K$. Let $N_{K/k}$ denote the norm map from $K$ to $k$. The cycle $\epsilon$ is determined by the property $N_{K/k}(K_\infty^\times(\tilde{c})^\times) = k_\infty^\times(\epsilon)^\times$. Put $\phi(\epsilon)^\times := \phi^\times \cap i_\infty^{-1}(k_\infty^\times(\epsilon)^\times)$, where $i_\infty : k^\times \to k_\infty^\times$ is the diagonal embedding. Denote by $V_f$ the set of finite places of $k$. Let $e(v)$ denote the ramification index of any place $w$ over $v \in V_f$.

Theorem 1.1. One has

$$\# \text{Cl}(K, \tilde{c})^G = \frac{\# \text{Cl}(k, \epsilon) \prod_{v \in V_f} e(v)}{[K : k][\phi(\epsilon)^\times : \phi(\epsilon)^\times \cap N_{K/k}(K^\times)]}.$$
When $\tilde{c} = \infty$, we get the restricted version of the formula stated in [2, p. 178]. When $\tilde{c} = \emptyset$, using an elementary fact
\[
\# \text{Cl}(k, \infty) = \frac{h(k) \cdot 2^{[\infty : \sigma(\infty^\times)]}}{\sigma^\times : \sigma(\infty^\times)\sigma^\times},
\]
we get the ordinary version of the formula stated in [2, p. 180].

2. Proof of Theorem 1.1

Define the norm ideal class group $N(K, \tilde{c})$ by
\[
(2.1) \quad N(K, \tilde{c}) := \frac{N_{K/k}(\mathbb{A}_K^\times)}{N_{K/k}(K^\times \hat{O}^\times K_\infty(\tilde{c})^\times)}.
\]
Consider the commutative diagram of two short exact sequences (by Hilbert’s Theorem 90)
\[
\begin{array}{ccccccc}
1 & \longrightarrow & \mathbb{A}_K^{1-\sigma} \cap U & \longrightarrow & U & \longrightarrow & N_{K/k}(U) & \longrightarrow & 1 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathbb{A}_K^{1-\sigma} & \longrightarrow & \mathbb{A}_K^\times & \longrightarrow & N_{K/k}(\mathbb{A}_K^\times) & \longrightarrow & 1,
\end{array}
\]
where $U = K^\times \hat{O}^\times K_\infty(\tilde{c})^\times$. The snake lemma gives the short exact sequence
\[
(2.3) \quad 1 \longrightarrow \text{Cl}(K, \tilde{c})^{1-\sigma} \longrightarrow \text{Cl}(K, \tilde{c}) \longrightarrow N(K, \tilde{c}) \longrightarrow 1
\]
as one has an isomorphism $\mathbb{A}_K^{1-\sigma}/(\mathbb{A}_K^{1-\sigma} \cap U) \simeq \text{Cl}(K, \tilde{c})^{1-\sigma}$. On the other hand we have the short exact sequence
\[
(2.4) \quad 1 \longrightarrow \text{Cl}(K, \tilde{c})^G \longrightarrow \text{Cl}(K, \tilde{c}) \longrightarrow \text{Cl}(K, \tilde{c})^{1-\sigma} \longrightarrow 1,
\]
which with (2.3) shows the following result.

Lemma 2.1. We have $\# \text{Cl}(K, \tilde{c})^G = \# N(K, \tilde{c})$.

Define
\[
\text{Cl}(k, c, \mathcal{D}) := \frac{\mathbb{A}_k^\times}{k^\times k_\infty(\sigma)^\times N_{K/k}(\hat{O}^\times)}.
\]

Lemma 2.2. The group $N(K, \tilde{c})$ is isomorphic to a subgroup $H \subset \text{Cl}(k, c, \mathcal{D})$ of index $[K : k]$.

Proof. Put $A := N_{K/k}(\mathbb{A}_K^\times)$, $B := N_{K/k}(K^\times \hat{O}^\times K_\infty(\tilde{c})^\times)$, $C := k^\times$ and $H := CA/CB$. The group $H$ is a subgroup in $\text{Cl}(k, c, \mathcal{D})$, which is of index $[K : k]$ by the global norm index theorem [1, p. 193]. One has $A \cap C = N_{K/k}(K^\times) \subset B$ by the Hasse norm theorem [1, p. 195]. The lemma follows from
\[
N(K, \tilde{c}) = A/B = A/(A \cap C)B \simeq CA/CB = H.
\]

Consider the exact sequence
\[
(2.5) \quad 1 \longrightarrow \frac{\sigma(\alpha)^\times}{\sigma(\alpha)^\times \cap N(\mathcal{D}^\times)} \longrightarrow \hat{\mathcal{O}}^\times \longrightarrow \text{Cl}(k, c, \mathcal{D}) \longrightarrow \text{Cl}(k, c) \longrightarrow 1.
\]
It is easy to see \( o(\mathfrak{c})^\times \cap N_{K/k}(\hat{\mathfrak{O}}^\times) = o(\mathfrak{c})^\times \cap N_{K/k}(K^\times) \) from the Hasse norm theorem. The local norm index theorem [1, p. 188, Lemma 4] gives
\[
\# \left( \frac{\mathfrak{d}^\times}{N(\hat{\mathfrak{O}}^\times)} \right) = \prod_{v\in V_f} e(v).
\]
Combining Lemma 2.2, (2.5) and (2.6) we get
\[
\#N(K, \tilde{\mathfrak{c}}) = \frac{\# \text{Cl}(k, \mathfrak{c}, \mathfrak{O})}{[K : k]} \prod_{v\in V_f} e(v).
\]
Theorem 1.1 follows from Lemma 2.1 and (2.7). 

**Remark 2.3.** We do not know whether \( \text{Cl}(K, \tilde{\mathfrak{c}})^G \) and \( N(K, \tilde{\mathfrak{c}}) \) are isomorphic as abelian groups or whether there is a natural bijection between them. When \( [K : k] = 2 \) and \( \# \text{Cl}(K, \tilde{\mathfrak{c}})^{1-\sigma} \) is odd, we show that there is a natural isomorphism
\[
\#N(K, \tilde{\mathfrak{c}}) = \frac{\# \text{Cl}(k, \mathfrak{c})}{\# \text{Cl}(K, \tilde{\mathfrak{c}})^{1-\sigma}}.
\]
The map \( 1 - \sigma : \text{Cl}(K, \tilde{\mathfrak{c}}) \to \text{Cl}(K, \tilde{\mathfrak{c}})^{1-\sigma} \) restricted to \( \text{Cl}(K, \tilde{\mathfrak{c}})^{1-\sigma} \) is the squared map \( \text{Sq} \), which is an isomorphism from our assumption. The inverse of \( \text{Sq} \) defines a section of (2.4), and hence an isomorphism \( \text{Cl}(K, \tilde{\mathfrak{c}}) \simeq \text{Cl}(K, \tilde{\mathfrak{c}})^G \oplus \text{Cl}(K, \tilde{\mathfrak{c}})^{1-\sigma} \). The assertion (2.8) then follows.

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